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Lie symmetries for integrable evolution equations on the lattice

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Abstract. In this work we show how to construct symmetries for the differential-difference equations associated with the discrete Schrödinger spectral problem. We find the whole set of symmetries which in the continuous limit go into the Lie point symmetries of the corresponding partial differential equation, i.e. the Korteweg–de Vries equation. Among these, of particular relevance, is the non-autonomous symmetry which, in the continuous limit, goes into the dilation symmetry for the corresponding equation. Unlike the continuous case, this symmetry turns out to be a master symmetry, thus belonging to the infinite-dimensional group of generalized symmetries.

1. Introduction

Nonlinear differential-difference equations are always very important in applications. They enter as models for many biological chains, are encountered frequently in queuing problems and as discretizations of field theories. So, both as themselves and as approximations of continuous problems, they play a very important role in many fields of mathematics, physics, biology and engineering.

Not many tools are available to solve such kinds of problems. Apart from a few exceptional cases the solution of nonlinear differential-difference equations can only be obtained by numerical calculations or by going to the continuous limit when the lattice spacing vanishes and the system is approximated by a continuous nonlinear partial differential equation. Exceptional cases are those equations which, in a way or another, are either linearizable or integrable via the solution of an associated spectral problem on the lattice. In such cases we can write down a denumerable set of exact solutions corresponding to symmetries of the nonlinear differential-difference equations. Such symmetries can either depend just on the independent variable and on the dependent variable in the generic point of the lattice n and are denoted as point symmetries or depend on the dependent field in neighbouring positions of the lattice and in this case we speak of generalized symmetries [1]. Any differential-difference equation can have point symmetries, but the existence of generalized symmetries is usually associated only with the integrable ones.

In the case of pure differential equations the difference between point and generalized symmetries is very clear [2, 3]. The classical definition of a point symmetry requires that the infinitesimal generators of the Lie point symmetry group depend only on the independent and dependent variables and not on their derivatives. This implies that the finite group of Lie point transformations which leaves the differential equation invariant is obtained by integrating a quasilinear partial differential equation of first order which can be solved generically on the characteristics. In the case of generalized symmetries, the infinitesimal generators also depend on the derivatives of the dependent fields with respect to the independent variables and thus the partial differential equation whose solution would give the group transformation is no longer of first order. Moreover, point symmetries form a closed finite group while generalized symmetries form an infinite-dimensional group.

In the case of difference equations, i.e. when at least one of the independent variables varies discretely, one can consider the intrinsic point symmetries as a direct counterpart of the Lie point symmetries [4, 5]. In this case the discrete independent variable is not changed by the transformation. The point transformations act only on the continuous variables, be they independent or dependent. However, the transformation may depend parametrically on the discrete variables and the infinitesimal generators of the symmetries, consequently, may have a non-trivial dependence on the discrete variable. As shown in [6] the symmetry group thus obtained is a subgroup of the symmetry group of the differential equations to which the differential-difference equation goes in the continuous limit.

In the case of linear difference equations (and consequently linear differential-difference equations) one can construct a sequence of linear operators which have the same symmetry algebra as that for the corresponding linear differential equation [7]. Consequently, one can think of extending the formalism introduced to construct intrinsic symmetries so as to include all symmetries that have as a continuous limit point symmetries [8]. This requires the construction of symmetries whose infinitesimal generators depend on the dependent variable computed at different points of the lattice. We will call such symmetries *extended point symmetries*. From the point of view of the definition applied in the case of differential equations, the extended point symmetries are generalized symmetries.

In [8] it was shown that in the evolutionary approach to symmetries, which unifies both point and generalized symmetries, one can recover the extended point symmetries for linear equations but only partially for a generic nonlinear difference equation. In a subsequent work it has been shown how one can find extended point symmetries in the case of the discrete Burgers equation, a nonlinear linearizable equation [9].

Here we consider integrable equations associated with the discrete Schrödinger spectral problem. We construct the symmetries for the given integrable nonlinear differential-difference equations by looking for commuting flows. To obtain the commuting flows we study the spectral transform, which associates to each evolution equation belonging to the class of evolutions compatible with the discrete Schrödinger spectral problem, an evolution of the spectral data. The study of commuting spectral data allows us to associate to any nonlinear differential-difference equation a class of symmetries. These symmetries are split into two categories: one in which the eigenvalues are not evolving in the group parameter space and thus are represented by autonomous generators, depending on the dependent variable in a finite set of different points of the lattice, i.e. pure generalized symmetries. There are a denumerable set of these symmetries. Then we have the case when the eigenvalue depends on the group parameter. In this case the infinitesimal symmetry generators depend on the discrete variable; such symmetries, in the continuous limit, give rise to point symmetries.

In section 2 we review the results on the discrete Schrödinger spectral problem and on the associated class of differential-difference equations. In section 3 we introduce the evolutionary infinitesimal symmetry generators and show how they can be constructed by building up flows (in the group parameters) commuting with the differential-difference equations. We then construct commuting flows in the space of the spectral data and write down the local evolutionary symmetry generators associated with the Toda lattice hierarchy [10] and its reduction, the Volterra hierarchy. In section 4 we carry out the continuous limit and prove that the extended point symmetries go over to point symmetries in the continuous limit which carries the Toda lattice into the Korteweg–de Vries equation. Section 5 is devoted to some concluding remarks.

2. Discrete Schrödinger spectral problem and its associated differential-difference equations

The discrete Schrödinger spectral problem [11–13] was studied a long time ago as it is associated with the well known Toda lattice equation, the first differential-difference equation which has been proved to be integrable. For the sake of completeness we present in this section the main results which are relevant for the construction of the symmetries.

The discrete Schrödinger spectral problem reads

$$\psi(n-1,t;\lambda) + B(n,t)\,\psi(n,t;\lambda) + A(n,t)\,\psi(n+1,t;\lambda) = \lambda\psi(n,t;\lambda) \tag{2.1}$$

where λ, which can be written as $\lambda = z + 1/z$, is a spectral parameter; $A(n,t)$ and $B(n,t)$ are two functions depending on the integer variable n and parametrically on a real variable t with the asymptotic conditions

$$\lim_{|n|\to\infty} A(n,t) - 1 = \lim_{|n|\to\infty} B(n,t) = 0. \tag{2.2}$$

To the discrete Schrödinger spectral problem we can associate the following class of nonlinear evolution equations:

$$\begin{pmatrix} \dot{A}(n,t) \\ \dot{B}(n,t) \end{pmatrix} = f_1(\mathcal{L},t)\begin{pmatrix} A(n,t)[B(n,t)-B(n+1,t)] \\ A(n-1,t)-A(n,t) \end{pmatrix}$$
$$+ f_2(\mathcal{L},t)\begin{pmatrix} A(n,t)[(2n+3)B(n+1,t)-(2n-1)B(n,t)] \\ B^2(n,t)-4+2[(n+1)A(n,t)-(n-1)A(n-1,t)] \end{pmatrix} \tag{2.3}$$

where by a dot we mean a partial derivative with respect to the t-variable and f_1 and f_2 are two arbitrary entire functions of the recursive operator $\mathcal{L}$:

$$\mathcal{L}\begin{pmatrix} P(n) \\ Q(n) \end{pmatrix} = \begin{pmatrix} P(n)B(n+1,t) + A(n,t)[Q(n)+Q(n+1)] + [B(n,t)-B(n+1,t)]S(n) \\ B(n,t)Q(n) + P(n) + S(n-1) - S(n) \end{pmatrix} \tag{2.4a}$$

with $S(n)$ the asymptotically bounded solution of the non-homogeneous first-order difference equation

$$S(n+1) = \frac{A(n+1,t)}{A(n,t)}[S(n)-P(n)]. \tag{2.4b}$$

In correspondence with the non-local evolution equations (2.3) we have

$$\dot{\psi}(n,t;\lambda) = M\psi(n,t;\lambda) + \psi(n,t;\lambda)\Gamma(\lambda,t) \tag{2.5}$$

$$\dot{\lambda} = (\lambda^2-4)f_2(\lambda,t) \tag{2.6}$$

where (see [13])

$$M\psi(n,t;\lambda) = -f_1(\mathcal{O},t)A(n,t)\psi(n+1,t;\lambda) - f_2(\mathcal{O},t)\Bigg(n\psi(n-1,t;\lambda)$$
$$+\Bigg[nB(n,t) - \sum_{j=n+1}^{\infty} B(j,t)\Bigg]\psi(n,t;\lambda) - nA(n,t)\,\psi(n+1,t;\lambda)\Bigg) \tag{2.7}$$

with the operator $\mathcal{O}$ related to the action of $\mathcal{L}$ given by (2.4*a*) defined by

$$\mathcal{O}\varphi(n) = \varphi(n-1) + \left[B(n,t) - \sum_{j=n}^{\infty} Q(j)\right]\varphi(n) + \frac{A(n,t)}{A(n+1,t)}[A(n+1,t) + S(n+1)]\varphi(n+1). \tag{2.8}$$

The function $\Gamma(\lambda)$ is determined by the asymptotic behaviour of ψ.

Whenever $A(n,t)$ and $B(n,t)$ satisfy the boundary conditions (2.2), the spectrum of (2.1) consists of the unit circle in the z complex plane, $\mathbf{C}_1$, plus a finite number of isolated points z_j inside $\mathbf{C}_1$. The spectral data which allow us to recover uniquely the potentials $A(n,t)$ and $B(n,t)$ are given by

$$\{R(z,t), z \in \mathbf{C}_1; z_j, c_j, |z_j| < 1, j = 1, 2, \ldots, N\} \tag{2.9}$$

with the 'reflection' coefficient $R(z,t)$ defined, as usual, by

$$\psi(n,t;\lambda) \sim z^{-n} + R(z,t)z^n \qquad n \to +\infty \tag{2.10}$$

while the coefficients c_j are defined by the asymptotic behaviour of the bound state eigenfunctions and are related to the residues of the reflection coefficient $R(z,t)$ at the points z_j.

In correspondence with the compatible integrable evolution (2.3) of the fields $A(n,t)$, $B(n,t)$ we obtain an evolution of the spectrum. We have that

$$\Gamma(\lambda) = f_1(\lambda,t)z^{-1} - \mu^2 \frac{\mathrm{d}}{\mathrm{d}\lambda} f_2(\lambda,t) \qquad \mu = z^{-1} - z \tag{2.11}$$

$$\frac{\mathrm{d}R(z,t)}{\mathrm{d}t} = \mu f_1(\lambda,t)R(z,t) \tag{2.12}$$

where, from now on, $\mathrm{d}/\mathrm{d}y$ denotes a total derivative with respect to y for any variable y.

Let us remark that formally we can extend still further the class of equations (2.3) associated with the spectral problem (2.1) as

$$\begin{pmatrix} A(n,t)[(2n+3)B(n+1,t) - (2n-1)B(n,t)] \\ B^2(n,t) - 4 + 2[(n+1)A(n,t) - (n-1)A(n-1,t)] \end{pmatrix} = \mathcal{L}\begin{pmatrix} 2A(n,t) \\ B(n,t) \end{pmatrix} - 4\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (\mathcal{L}^2 - 4)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.13}$$

having taken $S(n) = -nA(n)$ as a solution for equation (2.4*b*) for $P(n) = A(n)$. Taking into account formula (2.13), equation (2.3) can be written as

$$\begin{pmatrix} \dot{A}(n,t) \\ \dot{B}(n,t) \end{pmatrix} = f_1(\mathcal{L},t)\begin{pmatrix} A(n,t)[B(n,t) - B(n+1,t)] \\ A(n-1,t) - A(n,t) \end{pmatrix} + f_3(\mathcal{L},t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.14}$$

where equation (2.3) is recovered by choosing $f_3(\mathcal{L},t) = f_2(\mathcal{L},t)(\mathcal{L}^2 - 4)$.

Whenever $B(n,t) = 0$, the class of equations associated with the spectral problem (2.1) reduces to the Volterra hierarchy. This is obtained by restricting the functions f_1 and f_2 to odd functions, i.e.

$$\dot{A}(n,t) = g_1(\tilde{\mathcal{L}},t)\{A(n,t)(A(n-1,t) - A(n+1,t))\} + g_2(\tilde{\mathcal{L}},t)\{A(n,t)[A(n,t) - (n-1)A(n-1,t) + (n+2)A(n+1,t) - 4]\} \tag{2.15}$$

where

$$\tilde{\mathcal{L}}P(n) = A(n,t)[P(n) + P(n+1) + S(n-1) - S(n+1)] \tag{2.16}$$

with $S(n)$ given by equation (2.4*b*). Due to the boundary condition introduced to solve equation (2.4*b*), equations (2.15) with the recursive operator (2.16) have the correct spectral data and are different from those considered in the usual literature. The spectral data are defined in the same way as for the unreduced case. Equation (2.6) becomes

$$\dot{\lambda} = (\lambda^2 - 4)g_2(\lambda^2, t)\lambda \tag{2.17}$$

and equation (2.12) now reads

$$\frac{\mathrm{d}R(z,t)}{\mathrm{d}t} = \mu\lambda g_1(\lambda^2, t)R(z,t). \tag{2.18}$$

As for the case of the Toda hierarchy, in this case we can also extend the class of non-isospectral terms in (2.14) by noting that

$$A(n,t)[A(n,t) - (n-1)A(n-1,t) + (n+2)A(n+1,t) - 4] = (\tilde{\mathcal{L}} - 4)A(n) \tag{2.19}$$

having taken $S(n) = -nA(n)$ as a solution for equation (2.4*b*) for $P(n) = A(n)$.

3. Symmetries

The classical theory of Lie [2, 3] tells us that the symmetries, i.e. the group of point transformations for differential equations, are obtained by exponentiating the infinitesimal generators. For the sake of simplicity of exposition we consider here just one ordinary differential equation of second order for one independent and one dependent variable but all the results are valid in more general cases. Defining the infinitesimal generators as

$$\hat{X} = \xi(x,u)\partial_x + \phi(x,u)\partial_u \tag{3.1}$$

where ξ and ϕ are the coefficients of the generators depending just on the dependent and independent variables, a given equation

$$F(x, u, u_x, u_{xx}) = 0 \tag{3.2}$$

admits the following one-parameter group of transformations:

$$\tilde{x} = \mathrm{e}^{\epsilon\hat{X}}x \qquad \tilde{u} = \mathrm{e}^{\epsilon\hat{X}}u \tag{3.3}$$

if

$$\mathrm{pr}^{(2)}\,\hat{X}F|_{F=0} = 0 \tag{3.4}$$

where by $\mathrm{pr}^{(2)}\,\hat{X}$ we mean the second prolongation of the vector field $\hat{X}$.

The group of transformations (3.3) can also be obtained by solving the system of differential equations

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\epsilon} = \xi(\tilde{x},\tilde{u}) \qquad \frac{\mathrm{d}\tilde{u}}{\mathrm{d}\epsilon} = \phi(\tilde{x},\tilde{u}) \tag{3.5}$$

with the initial conditions $\tilde{x}(\epsilon = 0) = x$, $\tilde{u}(\epsilon = 0) = u$.

Given the differential equation (3.2), equation (3.4) provides a set of overdetermined linear differential equations whose solution gives the infinitesimal generators and thus the group of transformations. It can be easily shown that the same class of transformations can be obtained by considering instead of the infinitesimal generator (3.1) the following operator:

$$\hat{X}_e = [\phi(x,u) - \xi(x,u)u_x]\partial_u \tag{3.6}$$

which corresponds to a transformation

$$\tilde{x} = x \qquad \tilde{u} = \mathrm{e}^{\epsilon \hat{X}_e} u \tag{3.7}$$

i.e. all the changes are contained in the evolution of the dependent variable. Thus this approach is called an evolutionary formalism. This case is, in principle, simpler as we are considering only transformations of the dependent variables. However, we pay a price as equations (3.5), which we have to solve to obtain the group of transformations, are now replaced by a quasilinear partial differential equation of first order, integrable on the characteristics

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\epsilon} = \frac{\partial \tilde{u}}{\partial \epsilon} = \phi(x, \tilde{u}) - \xi(x, \tilde{u}) \frac{\partial \tilde{u}}{\partial x} \qquad \tilde{u}(\epsilon = 0) = u. \tag{3.8}$$

The main advantage of the use of the evolutionary formalism for the construction of symmetries relies on the fact that, apart from the simplicity of the construction of the prolongation, it is easily extendible to more general transformations when the infinitesimal generators depend on the derivatives of the dependent variable with respect to the independent one, i.e. on the generalized symmetries. In such a case, we can define

$$\hat{X}_e = Q(x, u, u_x, u_{xx}, \ldots)\partial_u \tag{3.9}$$

and equation (3.4) reads

$$\left.\frac{\partial F}{\partial u} Q + \frac{\partial F}{\partial u_x} D_x Q + \frac{\partial F}{\partial u_{xx}} D_x^2 Q \right|_{F=0} = 0 \tag{3.10}$$

where D_x is the total derivative operator in the x variable:

$$D_x = \partial_x + u_x \partial_u + u_{xx} \partial_{u_x} + \cdots. \tag{3.11}$$

The group transformations are obtained by solving the partial differential equations:

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\epsilon} = Q(x, \tilde{u}, \tilde{u}_x, \tilde{u}_{xx}, \ldots) \tag{3.12}$$

with the boundary condition $\tilde{u}(x, \epsilon = 0) = u(x)$, where $u(x)$ is the solution of equation (3.2). It is easy to show that equation (3.10) can be obtained by considering the compatibility of equation (3.12) with equation (3.2) written out for u substituted by $\tilde{u}$. This implies that the admissible symmetries of the given equation (3.2) are obtained as flows in the variable ϵ compatible with the equation under study. This point of view allows us not only to extend the class of symmetries under study from point to generalized, but also the class of equations from purely differential to functional equations and, in particular, differential-difference or difference equations. In fact, it is possible, in principle, to construct compatible flows for any kind of functional equations. If these compatible flows exist then we have the infinitesimal generators of the symmetries. To construct the symmetries, i.e. the group of transformations, we need to solve the equivalent of equation (3.12). It is easy to show that the solution of equation (3.12) we obtain by power-series expansion, i.e. setting

$$\tilde{u}(x, \epsilon) = \sum_{j=0}^{\infty} \epsilon^j \tilde{u}_j(x) \qquad \tilde{u}_0(x) = u(x) \tag{3.13}$$

is equivalent to the exponential representation

$$\tilde{u}(x, \epsilon) = \exp(\epsilon \hat{X}_\epsilon) u(x) \tag{3.14}$$

thus proving the validity of equation (3.7) for generalized symmetries.

Let us, for example, consider the following difference equation:

$$F(x, u(x), \Delta_x u(x), \Delta_x^2 u(x)) = 0. \tag{3.15}$$

where

$$\Delta_x u(x) = \frac{u(x+\sigma) - u(x)}{\sigma} \tag{3.16}$$

where σ is a constant parameter such that as $\sigma \to 0$, $\Delta_x u(x) = \frac{\mathrm{d}}{\mathrm{d}x} u(x)$.

Let us introduce an infinitesimal evolutionary generator whose coefficient Q may depend on x, σ and $u(x)$ at various points in the lattice. Then

$$\frac{\mathrm{d}u}{\mathrm{d}\epsilon} = Q(x, \sigma, \{u(x + k\sigma, \epsilon)\}_{k=-a}^{b}) \tag{3.17}$$

where a and b are two integers defining the span of the symmetry. Q is an infinitesimal coefficient for a symmetry of equation (3.15), if equation (3.15) and equation (3.17) are compatible, i.e.

$$F_{,u(x)} Q + F_{,\Delta_x u} \Delta_x^T Q + F_{,\Delta_x^2 u} (\Delta_x^T)^2 Q|_{F=0} = 0 \tag{3.18}$$

where by $\Delta_x^T Q$ we mean the total variation of Q with respect to x given by

$$\begin{aligned}\Delta_x^T Q\big(x, \sigma, \{u(x + k\sigma, \epsilon)\}_{k=-a}^{b}\big) &= \frac{1}{\sigma}\big[Q\big(x+\sigma, \sigma, \{u(x + k\sigma, \epsilon)\}_{k=-a+1}^{b+1}\big) \\ &\quad - Q(x, \sigma, \{u(x + k\sigma, \epsilon)\}_{k=-a}^{b}\big].\end{aligned} \tag{3.19}$$

Given equation (3.15), equation (3.18) is an equation for Q, i.e. the symmetry, and vice versa, given the symmetry (3.17), equation (3.18) defines the class of equations (3.15) which have that symmetry. As shown in [8], it is not easy to find a non-trivial solution of equation (3.18) for a general discrete equation unless one restricts the class of symmetry one is looking for, i.e. the form of Q.

Here, in the following we show that in the case of integrable differential-difference equations we can always construct all the symmetries using their integrability properties. In fact, it is well known that different flows associated with different isospectral t-evolutions for a given spectral problem, be they discrete or continuous, are commuting among themselves [14]. Thus considering the group parameter as a t-variable, we find that for a given isospectral equation (2.3):

$$\begin{pmatrix} \dot{A}(n,t) \\ \dot{B}(n,t) \end{pmatrix} = f_1(\mathcal{L}) \begin{pmatrix} A(n,t)[B(n,t) - B(n+1,t)] \\ A(n-1,t) - A(n,t) \end{pmatrix} \tag{3.20}$$

we can associate the following symmetries:

$$\begin{pmatrix} A_{\epsilon_k}(n,t) \\ B_{\epsilon_k}(n,t) \end{pmatrix} = \mathcal{L}^k \begin{pmatrix} A(n,t;\epsilon_k)[B(n,t;\epsilon_k) - B(n+1,t;\epsilon_k)] \\ A(n-1,t;\epsilon_k) - A(n,t;\epsilon_k) \end{pmatrix} \tag{3.21}$$

for any integer k†. The existence of non-isospectral deformations implies that we can construct other symmetries. As we are not interested in non-local symmetries, we have to limit ourselves

† An easy proof of this proposition can be given by taking into account the spectral transform. In fact, in this case to the two flows (3.20) and (3.21) there corresponds the following evolution of the spectrum: $\mathrm{d}R(z,t,\epsilon_k)/\mathrm{d}t = \mu f_1(\lambda) R(z,t,\epsilon_k)$; $\mathrm{d}R(z,t,\epsilon_k)/\mathrm{d}\epsilon_k = \mu\lambda^k R(z,t,\epsilon_k)$ which are easily shown to commute.

to f_2 constant in equation (2.3). So we can associate with equation (3.20) the following symmetries:

$$\begin{pmatrix} A_\epsilon(n,t;\epsilon) \\ B_\epsilon(n,t;\epsilon) \end{pmatrix} = tf_3(\mathcal{L}) \begin{pmatrix} A(n,t;\epsilon)[B(n,t;\epsilon) - B(n+1,t;\epsilon)] \\ A(n-1,t;\epsilon) - A(n,t;\epsilon) \end{pmatrix} + \alpha \begin{pmatrix} A(n,t;\epsilon)[(2n+3)B(n+1,t;\epsilon) - (2n-1)B(n,t;\epsilon)] \\ B^2(n,t;\epsilon) - 4 + 2[(n+1)A(n,t;\epsilon) - (n-1)A(n-1,t;\epsilon)) \end{pmatrix} \tag{3.22a}$$

with α a group parameter, $f_3(\mathcal{L})$ given by

$$f_3(\mathcal{L}) = \alpha\left[(\mathcal{L}^2 - 4)\frac{\partial f_1}{\partial \mathcal{L}}(\mathcal{L}) + \mathcal{L} f_1(\mathcal{L})\right] \tag{3.22b}$$

and

$$\lambda_\epsilon = \alpha\mu^2. \tag{3.22c}$$

We prove the validity of equation (3.22*a*) in the same way as we did for equation (3.21), by requiring that the spectrum of the t-flow and the ϵ-flow commute. By direct calculation it is easy to prove that

$$\frac{\mathrm{d}R(z,t;\epsilon)}{\mathrm{d}\epsilon} = \mu\alpha t\big[\lambda f_1(\lambda) + (\lambda^2 - 4) f_{1,\lambda}(\lambda)\big] R(z,t;\epsilon) \tag{3.23}$$

and (2.12) are compatible. As one can show from equation (2.13), one can add two further terms to equation (3.22*a*) given by

$$\beta \begin{pmatrix} 2A(n,t;\epsilon) \\ B(n,t;\epsilon) \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.24}$$

However, whenever γ and β are different from zero, we have not been able to write down a general expression for $f_3(\mathcal{L})$ as a function of γ, β and $f_1(\mathcal{L})$. In such a case the value of $f_3(\mathcal{L})$ has to be calculated directly for each equation in the hierarchy. The equation for the ϵ-evolution of λ is given by

$$\lambda_\epsilon = \alpha\mu^2 + \beta\lambda + \gamma. \tag{3.25}$$

As an example let us consider the particularly interesting case of $f_1(\lambda) = 1$, which corresponds to the case of the Toda lattice. In this case equation (3.20) is nothing else but the system representation of the Toda lattice and with the definition

$$B(n,t) = \dot{u}(n,t) \qquad A(n,t) = \mathrm{e}^{u(n,t)-u(n+1,t)} \tag{3.26}$$

it reduces to the well known Toda equation

$$\ddot{u}(n,t) = \mathrm{e}^{u(n-1,t)-u(n,t)} - \mathrm{e}^{u(n,t)-u(n+1,t)}. \tag{3.27}$$

In this case equation (3.22*a*), including the terms (3.24), can be written as

$$\begin{pmatrix} A_\epsilon(n,t;\epsilon) \\ B_\epsilon(n,t;\epsilon) \end{pmatrix} = \alpha \begin{pmatrix} A(n,t;\epsilon)[(2n+3)B(n+1,t;\epsilon) - (2n-1)B(n,t;\epsilon)] \\ B^2(n,t;\epsilon) - 4 + 2[(n+1)A(n,t;\epsilon) - (n-1)A(n-1,t;\epsilon) \end{pmatrix} + \beta \begin{pmatrix} 2A(n,t;\epsilon) \\ B(n,t;\epsilon) \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \beta t \begin{pmatrix} \dot{A}(n,t;\epsilon) \\ \dot{B}(n,t;\epsilon) \end{pmatrix} + \alpha t \begin{pmatrix} A(n,t;\epsilon)[A(n-1,t;\epsilon) - A(n+1,t;\epsilon) + B^2(n,t;\epsilon) - B^2(n+1,t;\epsilon)] \\ A(n-1,t;\epsilon)(B(n-1,t;\epsilon) + B(n,t;\epsilon)) - A(n,t;\epsilon)(B(n,t;\epsilon) + B(n+1,t;\epsilon)) \end{pmatrix} \tag{3.28}$$

which provides three different infinitesimal generators:

$$\hat{X}_1 = \partial_{B(n,t)} \tag{3.29a}$$

$$\hat{X}_2 = [t\dot{A}(n,t) + 2A(n,t)]\partial_{A(n,t)} + [t\dot{B}(n,t) + B(n,t)]\partial_{B(n,t)} \tag{3.29b}$$

$$\begin{aligned}\hat{X}_3 = \{&t[A(n-1,t) - A(n+1,t) + B^2(n,t) - B^2(n+1,t)]\\ &+(2n+3)B(n+1,t) - (2n-1)B(n,t)\}A(n,t)\partial_{A(n,t)}\\ &+\{t[A(n-1,t)(B(n-1,t) + B(n,t)) - A(n,t)(B(n,t)\\ &+B(n+1,t))] + B^2(n,t) - 4 + 2[(n+1)A(n,t) - (n-1)A(n-1,t)]\}\partial_{B(n,t)}.\end{aligned} \tag{3.29c}$$

The symmetries $\hat{X}_1$ and $\hat{X}_2$ correspond to intrinsic point symmetries as there is no dependence on shifted variables, while $\hat{X}_3$ is a dilation-like symmetry, an extended point symmetry. Another intrinsic Lie point symmetry can be obtained from (3.21) for $k = 0$ and reads

$$\hat{X}_4 = \dot{A}(n,t)\partial_{A(n,t)} + \dot{B}(n,t)\partial_{B(n,t)}. \tag{3.29d}$$

Using (3.26) we can transform (3.29) in the form of symmetries for the Toda lattice equation (3.27). It is more convenient to start from (3.20) and (3.28) using the evolution of $B(n,t)$ with respect to the group parameter and the relation of $A(n,t)$, $B(n,t)$ to $u(n,t)$ given by (3.26). In such a case we have

$$\begin{aligned}\dot{u}_\epsilon(n,t) = \delta\ddot{u}(n,t) + 2\beta[\dot{u}(n,t) + t\ddot{u}(n,t)] + \gamma + \alpha\{&t\mathrm{e}^{u(n-1,t)-u(n,t)}[\dot{u}(n-1,t) + \dot{u}(n,t)]\\ &-t\mathrm{e}^{u(n,t)-u(n+1,t)}[\dot{u}(n,t) + \dot{u}(n+1,t)] + \dot{u}^2(n,t) - 4\\ &+2(n+1)\mathrm{e}^{u(n,t)-u(n+1,t)} - 2(n-1)\mathrm{e}^{u(n-1,t)-u(n,t)}\}.\end{aligned}$$

For $\alpha = 0$ we can integrate with respect to the t-variable and using the ϵ-evolution of $A(n,t,\epsilon)$ given by (3.28) we obtain

$$u_\epsilon(n,t) = \delta\dot{u}(n,t) + \beta(t\dot{u}(n,t) - 2n) + \gamma t + \omega \tag{3.30}$$

the four-dimensional intrinsic Lie point group of the Toda lattice equation [5, 6].

For $\beta = \delta = \gamma = 0$ and $\alpha = 1$ taking into account the evolution for $A(n,t;\epsilon)$ in equation (3.28) we obtain

$$\begin{aligned}u_\epsilon(n,t) - u_\epsilon(n+1,t) = t[&\mathrm{e}^{u(n-1,t)-u(n,t)} - \mathrm{e}^{u(n+1,t)-u(n+2,t)} + \dot{u}(n,t)^2 - \dot{u}(n+1,t)^2]\\ &+\dot{u}(n+1,t)(2n+3) - \dot{u}(n,t)(2n-1)\end{aligned}$$

and by 'integration' with respect to the discrete variable

$$\begin{aligned}u_\epsilon(n,t) = t[&\mathrm{e}^{u(n-1,t)-u(n,t)} + \mathrm{e}^{u(n,t)-u(n+1,t)} + \dot{u}^2(n,t) - 2]\\ &-\dot{u}(n,t)(2n-1) + 2\sum_{j=n+1}^{\infty}\dot{u}(j,t).\end{aligned} \tag{3.31}$$

Unlike equation (3.30), equation (3.31) corresponds to an extended Lie point symmetry and it is non-local.

For the sake of completeness, we will write down the algebra satisfied by the symmetry operators $\hat{X}_1$, $\hat{X}_2$, $\hat{X}_3$, $\hat{X}_4$ for the Toda lattice expressed in a system form in (3.29). To do so, we have, following [8], to apply the prolonged symmetry operator and then to project it into the $\{A_n, B_n\}$ space. The resulting commutation relations which are different from zero are

$$[\hat{X}_1, \hat{X}_2] = \hat{X}_1 \qquad [\hat{X}_1, \hat{X}_3] = 2\hat{X}_2 \qquad [\hat{X}_2, \hat{X}_3] = 8\hat{X}_1 + \hat{X}_3 \qquad [\hat{X}_2, \hat{X}_4] = \hat{X}_4$$

and $\hat{X}_3$ with $\hat{X}_4$ give rise to a higher symmetry, showing that $\hat{X}_3$ is a master symmetry [15]. Thus, whenever $\hat{X}_3$ and $\hat{X}_4$ are both present, we have an infinite-dimensional algebra. A closed Lie algebra can be obtained by considering either $\{\hat{X}_1, \hat{X}_2, \hat{X}_3\}$ or $\{\hat{X}_1, \hat{X}_2, \hat{X}_4\}$. Considering now the Toda lattice equations (3.27), the symmetry generators (3.30) and (3.31) read

$$\begin{aligned}
X_1 &= \dot{u}(n,t)\partial_{u(n,t)} \\
X_2 &= (t\dot{u}(n,t) - 2n)\partial_{u(n,t)} \\
X_3 &= t\partial_{u(n,t)} \\
X_4 &= \partial_{u(n,t)} \\
X_5 &= \Big\{ t[\mathrm{e}^{u(n-1,t)-u(n,t)} + \mathrm{e}^{u(n,t)-u(n+1,t)} + \dot{u}^2(n,t) - 2] \\
&\qquad -(2n-1)\dot{u}(n,t) + 2\sum_{j=n+1}^{\infty} \dot{u}(j,t) \Big\} \partial_{u(n,t)}
\end{aligned} \tag{3.32}$$

and the resulting commutation relations which are different from zero are

$$[X_1, X_2] = -X_1 \qquad [X_1, X_3] = -X_4 \qquad [X_2, X_3] = -X_3$$
$$[X_2, X_5] = 4X_3 + X_5 \qquad [X_3, X_5] = 2X_2.$$

The commutation relation between X_1 and X_5 gives a new symmetry,

$$X_6 = \mathrm{e}^{u(n-1,t)-u(n,t)} + \mathrm{e}^{u(n,t)-u(n+1,t)} + \dot{u}^2(n,t) - 2$$

which does not form a closed Lie algebra together with the other symmetries $(X_1, \ldots, X_5)$.

Let us now consider the case of the Volterra hierarchy. Given an equation of this hierarchy:

$$\dot{A}(n,t) = g_1(\tilde{\mathcal{L}})\big\{A(n,t)[A(n-1,t) - A(n+1,t)]\big\} \tag{3.33}$$

we can construct the infinite set of symmetries parametrized by an integer index k, obtained by considering the isospectral commuting flows:

$$A_{\epsilon_k}(n,t;\epsilon_k) = (\tilde{\mathcal{L}})^k\big\{A(n,t;\epsilon_k)[A(n-1,t;\epsilon_k) - A(n+1,t;\epsilon_k)]\big\} \tag{3.34}$$

and by considering the non-isospectral terms:

$$\begin{aligned}
A_\epsilon(n,t;\epsilon) &= \alpha A(n,t;\epsilon)[A(n,t;\epsilon) - (n-1)A(n-1,t;\epsilon) + (n+2)A(n+1,t;\epsilon) - 4] \\
&\quad + tg_3(\tilde{\mathcal{L}})\big\{A(n,t;\epsilon)[A(n-1,t;\epsilon) - A(n+1,t;\epsilon)]\big\}
\end{aligned} \tag{3.35a}$$

with

$$g_3(\tilde{\mathcal{L}}) = \alpha\left[(\tilde{\mathcal{L}} - 2)g_1(\tilde{\mathcal{L}}) + \tilde{\mathcal{L}}(\tilde{\mathcal{L}} - 4)\frac{\mathrm{d}g_1}{\mathrm{d}\tilde{\mathcal{L}}}(\tilde{\mathcal{L}})\right] \tag{3.35b}$$

and

$$\lambda_\epsilon = \tfrac{1}{2}\alpha\lambda\mu^2. \tag{3.35c}$$

From equation (2.19) one can add one further term to equation (3.35*a*) given by $\beta A(n,t;\epsilon)$. Moreover, whenever β is different from zero the expression of $g_3(\tilde{\mathcal{L}})$ cannot be written down explicitly in terms of $g_1(\tilde{\mathcal{L}})$ in the general case. In such a case the value of $g_3(\tilde{\mathcal{L}})$ has to be calculated directly for each equation in the hierarchy. The equation for the ϵ-evolution of λ is given by

$$\lambda_\epsilon = \tfrac{1}{2}\alpha\lambda\mu^2 + \tfrac{1}{2}\beta\lambda. \tag{3.36}$$

In the particular case of the Volterra equation, corresponding to $g_1(\tilde{\mathcal{L}}) = 1$, apart from those symmetries given by equation (3.34), we have

$$A_\epsilon(n,t;\epsilon) = \beta\{t\dot{A}(n,t;\epsilon) + A(n,t;\epsilon)\} + \alpha A(n,t;\epsilon)\{A(n,t;\epsilon) - (n-1)A(n-1,t;\epsilon) +(n+2)A(n+1,t;\epsilon) - 4 + t[A(n-1,t;\epsilon)(A(n-2,t;\epsilon) +A(n-1,t;\epsilon) + A(n,t;\epsilon) - 4) - A(n+1,t;\epsilon)(A(n+2,t;\epsilon) +A(n+1,t;\epsilon) + A(n,t;\epsilon) - 4)]\}. \tag{3.37}$$

The case when $\alpha = 0$ gives an intrinsic Lie point symmetry $\hat{X}_1 = [t\dot{A}(n,t) + A(n,t)]\partial_{A(n,t)}$, while when $\beta = 0$ we get an extended dilation-like Lie point symmetry, with generator

$$\hat{X}_2 = A(n,t)\{A(n,t) - (n-1)A(n-1,t) + (n+2)A(n+1,t) - 4 +t[A(n-1,t)(A(n-2,t) + A(n-1,t) + A(n,t) - 4) -A(n+1,t)(A(n+2,t) + A(n+1,t) + A(n,t) - 4)]\}\partial_{A(n,t)}. \tag{3.38}$$

A third intrinsic symmetry is given by $\hat{X}_3 = \dot{A}(n,t)\partial_{A(n,t)}$, while the remaining symmetries are generalized. It is also interesting to consider explicitly the case of a higher Volterra equation which has as a direct continuous limit the Korteweg–de Vries equation [6], corresponding to $g_1(\tilde{\mathcal{L}}) = \tilde{\mathcal{L}} - 4$ and given by

$$\dot{A}(n,t) = A(n,t)\{A(n-1,t)[A(n,t) + A(n-1,t) + A(n-2,t) - 6] -A(n+1,t)[A(n,t) + A(n+1,t) + A(n+2,t) - 6]\}. \tag{3.39}$$

In this case the lowest-order symmetries from the isospectral ones (3.34) and the non-isospectral ones (3.35*a*) give

$$\hat{X}_1 = A(n,t)[A(n-1,t) - A(n+1,t)]\partial_{A(n,t)} \tag{3.40a}$$

$$\hat{X}_2 = \dot{A}(n,t)\partial_{A(n,t)} \tag{3.40b}$$

$$\hat{X}_3 = A(n,t)\{1 + 2t[A(n-1,t)(A(n,t) + A(n-1,t) +A(n-2,t) - 3) - A(n+1,t)(A(n,t) +A(n+1,t) + A(n+2,t) - 3)]\}\partial_{A(n,t)} \tag{3.40c}$$

$$\hat{X}_4 = A(n,t)\{A(n,t) - (n-1)A(n-1,t) + (n+2)A(n+1,t) - 4 \; 2t[A(n-1,t)(A(n-2,t)(A(n-3,t) + A(n-2,t) + A(n-1,t)) +(A(n-2,t) + A(n-1,t) + A(n,t))(A(n-1,t) + A(n,t) - 7) + 12) -A(n+1,t)(A(n+2,t)(A(n+3,t) + A(n+2,t) +A(n+1,t)) + (A(n+2,t) + A(n+1,t) +A(n,t))(A(n+1,t) + A(n,t) - 7) + 12)]\}\partial_{A(n,t)}. \tag{3.40d}$$

Let us notice that equation (3.39) has only one intrinsic Lie point symmetry $\hat{X}_2$. All the other symmetries we wrote down are extended. Apart from the one we write down there are an infinite number of generalized ones obtained by choosing $k > 2$ in (3.34).

For the higher Volterra equation (3.39), the commutation relations among the symmetry operators $\hat{X}_1$, $\hat{X}_2$, $\hat{X}_3$, $\hat{X}_4$ which are different from zero are:

$$[\hat{X}_1, \hat{X}_3] = -\hat{X}_1 \qquad [\hat{X}_1, \hat{X}_4] = -2\hat{X}_1 - \hat{X}_2$$

$$[\hat{X}_2, \hat{X}_3] = -2(3\hat{X}_1 + \hat{X}_2) \qquad [\hat{X}_3, \hat{X}_4] = 4\hat{X}_3 + \hat{X}_4$$

while the commutator of $\hat{X}_2$ and $\hat{X}_4$ give a higher symmetry, thus showing that $\hat{X}_4$ is a master symmetry. A closed Lie algebra can be obtained by considering just $\{\hat{X}_1, \hat{X}_2, \hat{X}_3\}$.

4. Continuous limits: the Toda and higher Volterra equations versus the Korteweg–de Vries equation

In this section we complete the work started in [6] showing that the extra symmetries we wrote down in the previous section are exactly those necessary to complete the description of the point symmetries of the Korteweg–De Vries equation. One could consider here the limit of the whole hierarchies, be it the Toda lattice hierarchy or the Volterra one, however, this is beyond the scope of this paper and partly, concerning the equations, it has been done elsewhere [16].

Let us start from the Toda lattice (3.27). Its continuous limit into the potential Korteweg–de Vries equation

$$v_{x\tau} = v_{xxxx} + 6v_x v_{xx} \tag{4.1}$$

is obtained by setting [6]

$$u(n,t) = -\tfrac{1}{2}\Delta v(x,\tau) \qquad x = (n-t)\Delta \qquad \tau = -\tfrac{1}{24}\Delta^3 t. \tag{4.2}$$

The symmetries (3.30) give the translational and the Galilean boost symmetries. Let us now analyse the symmetry (3.28). By taking into account the definitions (4.2), considering $X_5 - 2X_3$, we get the following continuous limit:

$$v_\epsilon(x,\tau) = xv_x(x,\tau) + 3\tau v_\tau(x,\tau) + v(x,\tau) \tag{4.3}$$

the dilation symmetry of the potential Korteweg–de Vries equation. The symmetry $X_6 - 2X_4 - 2X_1$ gives in the continuous limit the time translation.

Let us now consider the case of the higher Volterra equation given by (3.39). In this case

$$A(n,t) = 1 + \Delta^2 q(x,\tau) \qquad x = n\Delta \qquad \tau = -2\Delta^3 t \tag{4.4}$$

reduces (3.39) to the Korteweg–de Vries equation

$$q_\tau = q_{xxx} + 6qq_x. \tag{4.5}$$

The only intrinsic Lie point symmetry of (3.39) is given by (3.40*b*) and in the continuous limit provides $X = q_\tau \partial_q$ (as already shown in [6]). Equation (3.40*a*) gives $X = q_x\partial_q$, the space translation. Equation (3.40*c*) gives $X = (1 + 6\tau q_x)\partial_q$, the Galilean boost and equation (3.40*d*) gives $X = (xq_x + 2q + 3\tau q_\tau)\partial_q$.

5. Conclusions

In this paper we have shown how, using the integrability properties of the Toda lattice hierarchy and its subhierarchy, the Volterra hierarchy, we have been able to construct for them a set of symmetries whose continuous limits are point symmetries. In the discrete case these symmetries are not all intrinsic Lie point symmetries; some of them are extended Lie point symmetries and are associated with non-isospectral deformations of the underlying spectral problem. However, in the examples we have considered here, the symmetry which goes in the continuous limit into a dilation, turns out to be a master symmetry, thus generating all higher symmetries, Thus a closed Lie algebra for discrete equations does not contain dilations. Dilations seem to be an index of integrability. This result is clearly a step towards the comprehension of the structure of symmetries for difference equations and is the reason why in the case of non-integrable nonlinear equations [8] one has not been able to obtain extended Lie point symmetries.

Lie point symmetries for differential equations as well as intrinsic Lie point symmetries for discrete equations are characterized by the fact that one can use them to perform symmetry reduction and provide explicit solutions. In the case of generalized symmetries or extended Lie point symmetries, we are faced with nonlinear equations, sometimes not easily reducible to ordinary difference equations and surely not expressible in terms of known transcendents or functions.

In the case of integrable differential equations, symmetry reduction provides Painlevé transcendents. In the case of discrete equations, discrete Painlevé transcendents have been obtained by using other techniques (see, for instance, the articles on this topic contained in section 5 of [17] and references therein) as only a few symmetries were known and those usually very trivial ones. Some results in the direction of constructing discrete Painlevé transcendents by starting from integrable discrete equations have been obtained [18, 19] by considering reduction by t-invariance for non-isospectral discrete equations. Clearly, the construction presented here allows us to explain the approach considered before and will permit the construction of new Painlevé equations. Work along these lines is in progress.

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